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# MATHIEU FUNCTIONS OF GENERAL ORDER: CONNECTION FORMULAE, BASE FUNCTIONS AND ASYMPTOTIC FORMULAE

## IV. THE LIOUVILLE–GREEN METHOD APPLIED TO THE MATHIEU EQUATION

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CONTENTS



The methods described in part III and the formulae derived in part II are applied to the construction of a comprehensive set of asymptotic formulae relating to the Mathieu equation $y'' + (\lambda + 2h^2 \cos 2z)\, y = 0$ with real parameters. These comprise formulae both (*a*) for the auxiliary parameters and (*b*), in terms of exponential and circular functions, for the fundamental solution, a function of a complex variable, and the various pairs of real-variable base-functions, all introduced in part II. With the aid of these, together with connection formulae also obtained in part II, approximations can readily be obtained for Mathieu functions of various types, including in particular periodic functions.

Formulae for solutions are applicable on the half-strip $\{z:0 \leqslant \mathrm{Re}\, z \leqslant \frac{1}{2}\pi, \mathrm{Im}\, z \geqslant 0\}$, with the transition point of the differential equation which lies on its frontier removed, or in the case of real-variable solutions of the ordinary or modified equation, on the interval $[0, \frac{1}{2}\pi]$ or $[0, \infty]$ respectively, with the same qualification as for the half-strip when this is relevant. The formulae cover the full range of the parameters subject to $\lambda \neq \pm 2h^2$.

The $O$-terms providing error estimates are uniformly valid on any subdomain of the independent variable and parameters on which they remain bounded.

## 1. The Liouville transformation

The various formulae to be obtained, which are collected together in §6 below, are all derived by the method of part III, with the use of the basic equation III, (1.6). They include formulae for connection coefficients as well as formulae for solutions. The latter are all valid as follows:

(i) complex-variable solutions – on the fundamental region $\Omega$ (see (1.11) below);

(ii) real-variable solutions, ordinary equation – on $[0, \frac{1}{2}\pi]$;

(iii) real-variable solutions, modified equation – on $[0, \infty]$,

or in each case on a specified subdomain or subinterval. Formulae can be obtained in other domains or intervals by means of connection formulae or period relations, although in some cases the full domain of validity of the formulae is larger than that specified above.

The Mathieu equation written in the complex-variable form II, (1.4),

$$y'' + (\lambda + 2h^2 \cos 2z)\, y = 0, \tag{1.1}$$

has the form III, (1.1), with $u$ replaced by $h$ and

$$f(z) = \lambda'/h^2 + 2 \cos 2z, \quad g(z) = \lambda - \lambda',$$

$\lambda'/h^2$ and $\lambda - \lambda'$ being treated as parameters additional to $u = h$.

The redundant quantity $\lambda'$ is required in connection with approximations in terms of parabolic cylinder functions. It is introduced here so that certain formulae will be available for application in part V, but the natural choice if L.–G. approximations only are required is $\lambda' = \lambda$.

Let
$$\Delta(z) = -\tfrac{1}{4}h^{-2}(\lambda' + 2h^2 \cos 2z) \tag{1.2}$$

and define the new independent variable $\xi$ with derivative

$$\mathrm{d}\xi/\mathrm{d}z = 2[\Delta(z)]^{\frac{1}{2}}. \tag{1.3}$$

The quantity $\lambda'$ may depend on $h$ as well as on $\lambda$; it is required to satisfy uniformly

both
$$\lambda' - \lambda = O(1), \tag{1.4a}$$
and
$$\lambda' - \lambda = (h^2/\lambda')\, O(1). \tag{1.4b}$$

Throughout this paper, the expression $O(1)$ is used in its broadest sense to represent a *bounded* function of all relevant quantities over some specified domain; in (1.4*a*, *b*) these are $h$, $\lambda'$ subject only to $h \neq 0$. In the rest of this section, but only here, the parameters are regarded as arbitrary but fixed, and $O(1)$ is bounded as a function of $z$, only under this condition; subsequently, it is bounded as a function of the parameters also.

To determine a suitable integral of (1.3), observe that on the region $D = \{z:\mathrm{Im}\, z > M\}$, for some $M \geqslant 0$ depending on the parameters, $\mathrm{d}\xi/\mathrm{d}z$ has two disjoint analytic branches; a specific branch is selected by assigning the value $\arg \Delta(z) = 0$ when $z \in D$ and $\mathrm{Re}\, z = \frac{1}{2}\pi$, and by defining the square root accordingly. Then on $D$,

$$\mathrm{d}\xi/\mathrm{d}z = 2\sin z + (\sin z)^{-1}\, O(1) = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}z} + \mathrm{e}^{\mathrm{i}z}\, O(1),$$

and the indefinite integral can be determined so that

$$\xi = -2\cos z + o(1) = -\mathrm{e}^{-\mathrm{i}z} + o(1) \tag{1.5a}$$

uniformly on $D$ as $\mathrm{Im}\, z \to \infty$; also,

$$\Delta(z) \sim \sin^2 z \sim \tfrac{1}{4}\exp[-2\mathrm{i}(z - \tfrac{1}{2}\pi)] \tag{1.5b}$$

under the same conditions

This branch of $\xi$ is clearly a periodic function of $z$, with period $2\pi$; the map $z \to \xi$ shares with the map $z \to -2\cos z$ the following properties, the branches of $\arg \xi$ and of $\arg \cos z$ being appropriately chosen:

(i) for $z \in D$, $n \in \mathbf{Z}$, $\mathrm{Re}\, z = \frac{1}{2}n\pi \Rightarrow \arg \xi = (1 - \frac{1}{2}n)\,\pi$; (1.6a)

(ii) $\{z : 0 \leqslant \mathrm{Re}\, z \leqslant \frac{1}{2}\pi,\ \mathrm{Im}\, z > M\}$ is mapped one-to-one *into* $\{\xi : \pi \geqslant \arg \xi \geqslant 0\}$; (1.6b)

(iii) the transformation $z \to \tilde{z} = z + n\pi$ $(n \in \mathbf{Z})$ induces the transformations

$$\left.\begin{aligned} \xi \to \tilde{\xi} &= \xi\,\mathrm{e}^{-n\mathrm{i}\pi}, \\ \Delta(z) \to \Delta(\tilde{z}) &= \Delta(z)\,\mathrm{e}^{-2n\mathrm{i}\pi}, \end{aligned}\right\} \tag{1.6c}$$

the determination of $\arg \Delta(z)$ being significant in (1.3) and in (1.7a) below.

The image in the $\xi$-plane of the region $D$ is the exterior of a certain simple closed curve, symmetric about both axes.

The corresponding Liouville transformation applied to the Mathieu equation (1.1) gives the differential equation

$$\mathrm{d}^2 v/\mathrm{d}\xi^2 = \{h^2 + \psi(z)\}\, v, \tag{1.7}$$

with independent variable $\xi$, dependent variable $v$ given by

$$y = F(z)\, v \quad \text{where} \quad F(z) = [\Delta(z)]^{-\frac{1}{4}}, \tag{1.7a}$$

and $\psi(z)$ given by

$$\psi(z) = \frac{\lambda' - \lambda}{4\Delta(z)} + \frac{1}{8}\frac{\cos 2z}{[\Delta(z)]^2} - \frac{5}{64}\frac{\sin^2 2z}{[\Delta(z)]^3}. \tag{1.7b}$$

The form (1.7) with the formulae (1.7a, b) remains valid on all branches of the many-valued function obtained by analytic continuation of $\xi$ from the branch on $D$ defined above, $F(z)$ being continued concurrently. In this section, however, consideration will be restricted to the original branch on $D$.

For this branch, $\mathrm{Im}\, z \to \infty \Leftrightarrow |\xi| \to \infty$, and it follows readily from (1.5a, b) and (1.7b) that

$$\psi(z) = O(\xi^{-2}) \tag{1.8}$$

on $D$, so that on any path,

$$\mathrm{var} \int \psi(z)\, \mathrm{d}\xi = \mathrm{var}\, \{\xi^{-1}\}\, O(1). \tag{1.8a}$$

Now if $|\mathrm{Re}\, z| \leqslant \frac{3}{2}\pi - \delta$ $(\delta > 0)$ and $\mathrm{Im}\, z$ is sufficiently large there is a $\xi$-progressive path originating from $\infty\,\mathrm{e}^{\mathrm{i}\pi}$ in the $\xi$-plane – that is from $\infty\,\mathrm{i}$ in the $z$-plane – and terminating at $z$, which satisfies the conditions of lemma 1 of part III, §5 and its corollary. Hence by the theory of the L.–G. method as outlined in part III, there is a unique solution of (1.7) of the form

$$v = c\mathrm{e}^{h\xi}\,(1+\eta),$$

where $c$ is an arbitrary complex constant and

$$|\eta| \leqslant \exp\{h^{-1}\,\mathrm{var}\int \psi(z)\,\mathrm{d}\xi\}-1 = h^{-1}\xi^{-1}O(1)$$

uniformly as $\xi\to\infty$. Now by (1.6), arg $F(z) = -\frac{1}{4}\pi$ when $z\in D$ and Re $z = 0$; thus a solution $y_1(z)$ of (1.1) which is real and positive for sufficiently large Im $z$ and which is recessive as $z\to\infty\,\mathrm{i}$ is obtained by setting $c = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}$, when with the aid of (1.7 $a$) and (1.5 $a$, $b$),

$$y_1(z) \sim \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\,\mathrm{e}^{h\xi} \sim (\cos z)^{-\frac{1}{2}}\,\mathrm{e}^{-2h\cos z} \tag{1.9}$$

uniformly on $|\mathrm{Re}\, z| \leqslant \frac{3}{2}\pi-\delta\,(\delta > 0)$ as Im $z\to\infty$, in agreement with II, (3.1). From (1.6 $b$, $c$) it follows that for $n\in \boldsymbol{Z}$,

$$y_1(z-n\pi) \sim \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}-\frac{1}{2}n)}F(z)\,\exp\,[(-1)^{n+1}\,h\xi] \tag{1.10}$$

uniformly on $|\mathrm{Re}\, z-n\pi| \leqslant \frac{3}{2}\pi-\delta$ as Im $z\to\infty$; in particular, with $n = 1$, this gives

$$y_1(z-\pi) \sim -\mathrm{i}(\cos z)^{-\frac{1}{2}}\,\mathrm{e}^{2h\cos z},$$

in agreement with II (3.2). It must be emphasized that in these formulae, uniformity is with respect to $z$ only and does not extend to the parameters.

The next step is to extend $\xi$ by direct analytic continuation into the half-strip

$$\Omega = \{z: 0 \leqslant \mathrm{Re}\, z \leqslant \tfrac{1}{2}\pi,\ \mathrm{Im}\, z \geqslant 0\}, \tag{1.11}$$

which is a fundamental region for the symmetry group of the differential equation, this group being generated by the three maps $z\to z+\pi$, $z\to -z$, $z\to \bar{z}$. On $\Omega$ the map $z\to\xi$ is one-to-one, taking the forms illustrated in figures 3–5 (§6) for different parameter ranges.

It suffices to obtain estimates for the variation of the error control function $\int\psi(z)\,\mathrm{d}\xi$ on suitable paths in $\Omega$, for such estimates are applicable by symmetry to paths in any congruent half-strip, and a path not contained in a single half-strip can be decomposed into subarcs that are. It is necessary to distinguish several parameter ranges; all estimates obtained in the course of the calculations are uniform on the relevant range.

## 2. The variation of the e.c.f. on arbitrary paths in $\Omega$: $\lambda' \leqslant -2h^2$

In this range, which requires subdivision, the Mathieu equation has one transition point in the closed region $\Omega$, on its frontier at $z = z_0 = \mathrm{i}a$, where $a > 0$ is defined by $\lambda'+2h^2\cosh 2a = 0$; let $\xi_0$ denote the corresponding value of $\xi$.

### ($a$) *The case* $\lambda' < -4h^2$.

Estimates are needed for $\xi-\xi_0$ and for its reciprocal in terms of $z$; it will be shown that for $z\in\Omega$ and uniformly with respect to the parameters in the specified range,

$$[\xi-\xi_0]^{\pm 1} = O(1)\times\begin{cases}(\sinh 2a)^{\mp 1}\,[\Delta(z)]^{\pm\frac{3}{2}} & \text{if}\quad |\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a\\ [\Delta(z)]^{\pm\frac{1}{2}} & \text{if}\quad |\Delta(z)| \geqslant \frac{1}{2}\sinh^2 a \quad\text{and}\quad \mathrm{Im}\,(z-z_0) \geqslant -c,\end{cases} \tag{2.1}$$

where $c$ is any positive constant. The reciprocal nature of the estimates with opposite signs will be noticed, and the estimates on the two subdomains are consistent on the common frontier, though the $O$-factors are not continuous there. If $a \leqslant c$, the union of the two subregions of $\Omega$ defined by the inequalities in (2.1) is the complete region $\Omega$.

To establish this estimate, let $t = \mathrm{e}^{-\mathrm{i}z}\mathrm{e}^{-a}$; then $t = 1$ at $z_0$ and the region $\Omega$ maps one-to-one onto the region $\{t\colon -\frac{1}{2}\pi \leqslant \arg t \leqslant 0, |t| \geqslant \mathrm{e}^{-a}\}$. Also,

$$\text{(i)}\quad \Delta(z) = \tfrac{1}{4}\mathrm{e}^{2a}(1-t^2)\,(1-\mathrm{e}^{-4a}t^{-2}),$$

and since $\mathrm{e}^{-2a} \leqslant k < 1$, where $k$ is a constant,

$$[\Delta(z)]^{\pm 1} = [\mathrm{e}^{2a}(1-t^2)]^{\pm 1}\,O(1) \tag{2.2}$$

on $\Omega$; also

$$\text{(ii)}\quad \xi-\xi_0 = -\mathrm{e}^{a}\int_1 (1-t^{-2})^{\frac{1}{2}}\,(1-\mathrm{e}^{-4a}t^{-2})^{\frac{1}{2}}\,\mathrm{d}t, \tag{2.2a}$$

where the square roots have their principal values.

Now define
$$f(t) = -\int_1 (1-t^{-2})^{\frac{1}{2}}\,\mathrm{d}t = \mathrm{e}^{-a}\int_{\xi_0} (1-\mathrm{e}^{-4a}t^{-2})^{-\frac{1}{2}}\,\mathrm{d}\xi.$$

Then
$$\mathrm{e}^{a}f(t)-(\xi-\xi_0) = \int_{\xi_0} [(1-\mathrm{e}^{-4a}t^{-2})^{-\frac{1}{2}}-1]\,\mathrm{d}\xi. \tag{2.3}$$

It is easily verified that on $\Omega$,

$$|(1-\mathrm{e}^{-4a}t^{-2})^{-\frac{1}{2}}-1| \leqslant k_1 < 1,$$

and also that the image of $\Omega$ in the $\xi$-plane is a star domain with vertex $\xi_0$. Thus the integration can be made along paths on which $\arg(\xi-\xi_0)$ is constant, whence it follows that

$$|\mathrm{e}^{a}f(t)-(\xi-\xi_0)| \leqslant k_1|\xi-\xi_0|; \tag{2.4}$$

since $k_1 < 1$, this gives on $\Omega$:

$$[\mathrm{e}^{a}f(t)]^{\pm 1} = [\xi-\xi_0]^{\pm 1}\,O(1). \tag{2.4a}$$

Now by explicit integration it can be shown that, if $-\frac{1}{2}\pi \leqslant \arg t \leqslant 0$,

$$f(t) \sim \begin{cases} \frac{1}{3}(t^2-1)^{\frac{3}{2}} & \text{as} \quad t\to 1,\\ -t & \text{as} \quad t\to\infty,\\ \ln t & \text{as} \quad t\to 0; \end{cases} \tag{2.5}$$

since $f(t)$ is analytic and without zeros, except at $t = 0$ and $t = 1$, it follows from the first two of (2.5) that

$$[f(t)]^{\pm 1} = O(1)\times\begin{cases} (t^2-1)^{\pm\frac{3}{2}} & \text{if} \quad t \text{ and } t^{-1} \text{ are bounded},\\ t^{\pm 1} & \text{if} \quad (t^2-1)^{-1} \text{ and } t^{-1} \text{ are bounded}. \end{cases} \tag{2.5a}$$

It is easily seen that the alternative conditions in (2.5$a$) are satisfied uniformly on the respective subregions of $\Omega$ specified in (2.1); since $(\sinh a)^{\pm 1} = \mathrm{e}^{\pm a}O(1)$, the formula (2.1) can now be deduced from (2.2), (2.4$a$) and (2.5$a$).

By writing (1.7$b$) in the form

$$\psi(z) = \frac{1}{4}\,\frac{\lambda'-\lambda+\frac{1}{4}}{\Delta(z)}-\frac{3}{16}\,\frac{\cosh 2a}{[\Delta(z)]^2}-\frac{5}{64}\,\frac{\sinh^2 2a}{[\Delta(z)]^3}, \tag{2.6}$$

it now follows from (2.1) that, uniformly on each of the two subregions, and hence also on their union, namely the set $\{z\in\Omega\colon |t| \geqslant \mathrm{e}^{-c}\}$,

$$\psi(z) = [\xi-\xi_0]^{-2}\,O(1), \tag{2.7}$$



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whence, uniformly for paths in the same region,

$$\operatorname{var} \int \psi(z)\, \mathrm{d}\xi = \operatorname{var}\, [\xi-\xi_0]^{-1}\, O(1). \tag{2.7a}$$

Consideration of the remaining part of $\Omega$ is deferred until subsection (*c*) below.

(*b*) *The case* $-4h^2 \leqslant \lambda' \leqslant -2h^2$

The formula corresponding to (2.1) is

$$[\xi-\xi_0]^{\pm 1} = O(1) \times \begin{cases} (\sinh 2a)^{\mp 1}\, [\Delta(z)]^{\pm \frac{3}{2}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a \\ [\Delta(z)]^{\pm 1} & \text{if} \quad \frac{1}{2}\sinh^2 a \leqslant |\Delta(z)| \leqslant \frac{1}{4} \\ [\Delta(z)]^{\pm\frac{1}{2}} & \text{otherwise.} \end{cases} \tag{2.8}$$

If $\lambda' \neq -2h^2$, let $t = \sin \frac{1}{2}z/\sinh \frac{1}{2}a$; then $t = \mathrm{i}$ when $z = z_0$ and $\Omega$ is mapped one-to-one into the first quadrant in the $t$-plane. Also,

$$\Delta(z) = 4 \sinh^2 \tfrac{1}{2}a\, (t^2+1)\, (\cos^2 \tfrac{1}{2}z + \sinh^2 \tfrac{1}{2}a) \tag{2.9}$$

and
$$\xi-\xi_0 = 8 \sinh^2 \tfrac{1}{2}a \int_i (t^2+1)^{\frac{1}{2}}\, (1+\sec^2 \tfrac{1}{2}z \sinh^2 \tfrac{1}{2}a)^{\frac{1}{2}}\, \mathrm{d}t. \tag{2.10}$$

Now define
$$f(t) = 2\int_i (t^2+1)^{\frac{1}{2}}\, \mathrm{d}t,$$

so that
$$4 \sinh^2 \tfrac{1}{2}a\, f(t) = \int_{\xi_0} (1+\sec^2 \tfrac{1}{2}z \sinh^2 \tfrac{1}{2}a)^{-\frac{1}{2}}\, \mathrm{d}\xi.$$

Then by an argument similar to that of §2(*a*),

$$[\xi-\xi_0]^{\pm 1} = [\sinh^2 \tfrac{1}{2}a\, f(t)]^{\pm 1}\, O(1) \tag{2.11}$$

on $\Omega$, uniformly with respect to the parameters. Next, if $0 \leqslant \arg t \leqslant \frac{1}{2}\pi$, then

$$f(t) \sim \begin{cases} \dfrac{1}{3\mathrm{i}}\, (t^2+1)^{\frac{3}{2}} & \text{as} \quad t \to \mathrm{i}, \\ t^2+1 & \text{as} \quad t \to \infty; \end{cases}$$

since $f(t)$ is analytic and without zeros in this region, except at $t = \mathrm{i}$, it follows that

$$[f(t)]^{\pm 1} = O(1) \times \begin{cases} (t^2+1)^{\pm\frac{3}{2}} & \text{if} \quad t^2+1 \text{ is bounded,} \\ (t^2+1)^{\pm 1} & \text{if} \quad [t^2+1]^{-1} \text{ is bounded.} \end{cases} \tag{2.12}$$

Again, if $z \in \Omega$ and $\Delta(z)$, or equivalently $\sin z$, is bounded, $\cos^2 \frac{1}{2}z + \sinh^2 \frac{1}{2}a$ is bounded and bounded away from zero, whence from (2.9),

$$[\Delta(z)]^{\pm 1} = [\sinh^2 \tfrac{1}{2}a\, (t^2+1)]^{\pm 1}\, O(1).$$

Similarly, but less directly, it is found that if $z \in \Omega$ and $\Delta(z)$ is bounded away from zero, then

$$[\Delta(z)]^{\pm\frac{1}{2}} = [\sinh^2 \tfrac{1}{2}a\, (t^2+1)]^{\pm 1}\, O(1).$$

The formula (2.8) now follows from (2.12) and the fact that $(\cosh \frac{1}{2}a)^{\pm 1}$ is bounded.

By means of (2.6) and (2.8) it can be shown finally that the estimates (2.7) and (2.7*a*), obtained on a limited region in case (*a*), for $\psi(z)$ and the variation of the e.c.f., are now valid on the whole of $\Omega$. The analysis requires modification if $\lambda' = -2h^2$, when $a = 0$, but the same conclusion holds.

(*c*) *The combined range* $\lambda' \leqslant -2h^2$

The next objective is to establish the validity of the following formulae when $z \in \Omega$ and $\lambda' \leqslant -2h^2$ subject to the condition (1.4*b*).

(i) On the subdomain $R_1 = \{z \in \Omega\colon \mathrm{Im}\,(\xi - \xi_0) \geqslant -c_a\}$,

where $c_a > 0$, depends on $a$ and is defined below (2.16),

$$\psi(z) = [\xi - \xi_0]^{-2}\, O(1), \tag{2.13}$$

whence on any path in $R_1$,

$$\mathrm{var}\int \psi(z)\,\mathrm{d}\xi = \mathrm{var}\,\Psi_1(z)\,O(1), \tag{2.13a}$$

where $\Psi_1(z) = [\xi - \xi_0]^{-1}$;

If $-4h^2 \leqslant \lambda' < -2h^2$, $R_1$ is the whole of $\Omega$.

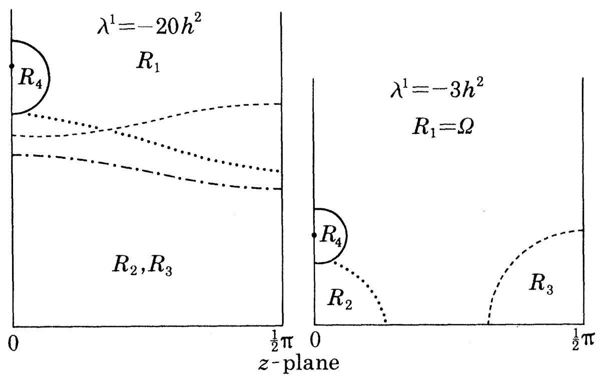


FIGURE 1. Subdomains of $\Omega$; their frontiers are identified thus: – · –, Fr $R_1$; · · ·, Fr $R_2$; - - -, Fr $R_3$; ——, Fr $R_4$.

(ii) On the subdomain $R_2 = \{z \in \Omega\colon |\sin^2 z| \leqslant \frac{1}{2}|\Delta(0)|\}$,

$$\psi(z)\,\mathrm{d}\xi/\mathrm{d}z = \mathrm{e}^{-2\mathrm{i}z}[\Delta(0)]^{-\frac{3}{4}}\,O(1), \tag{2.14}$$

whence on any path in $R_2$,

$$\mathrm{var}\int \psi(z)\,\mathrm{d}\xi = \mathrm{var}\,\Psi_2(z)\,O(1) \tag{2.14a}$$

where $\Psi_2(z) = (\mathrm{e}^{-2\mathrm{i}z} - 1)\,[\Delta(0)]^{-\frac{3}{4}}$.

(iii) On the subdomain $R_3 = \{z \in \Omega\colon |\cos^2 z| \leqslant \frac{1}{2}|\Delta(\frac{1}{2}\pi)|\}$,

$$\psi(z)\,\mathrm{d}\xi/\mathrm{d}z = \mathrm{e}^{-2\mathrm{i}z}[\Delta(\tfrac{1}{2}\pi)]^{-\frac{3}{4}}\,O(1), \tag{2.15}$$

whence on any path in $R_3$,

$$\mathrm{var}\int \psi(z)\,\mathrm{d}\xi = \mathrm{var}\,\Psi_3(z)\,O(1), \tag{2.15a}$$

where $\Psi_3(z) = (\mathrm{e}^{-2\mathrm{i}z} + 1)\,[\Delta(\frac{1}{2}\pi)]^{-\frac{3}{4}}$.

In (i) above, $$c_a = \sup\{-\mathrm{Im}\,(\xi - \xi_0)\colon z \in \Omega,\ z \notin R_2 \cap R_3\}. \tag{2.16}$$

The above subdomains of $\Omega$ are defined in such a way that $R_1 \cup R_2 = R_1 \cup R_3 = \Omega$; they are illustrated in figure 1, together with $R_4 = \{z \in \Omega\colon |\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a\}$ (see (2.1), (2.8)).

Since $R_1 = \Omega$ if $-4h^2 \leqslant \lambda' \leqslant -2h^2$, (2.13) follows in this case from (2.7). If on the other hand $\lambda' \leqslant -4h^2$, then $|t|^{-1}$, where $t$ is defined as in §2 (*a*), is uniformly bounded on $R_1$. For since $t_a = \inf\{|t| : z \in R_1\}$ is evidently a continuous function of $a$ for $a \geqslant 0$, it suffices to show that its reciprocal is bounded for all sufficiently large values of $a$; this is tedious rather than difficult and may be done by using the fact that when $a$ is large, $k_1$ in (2.4) may be chosen to be small. The details are omitted. On substituting $c = \ln \sup_{a \geqslant 0} \{|t_a|^{-1}\}$ in the formula for the domain of validity of (2.7), it follows that this last, and hence also (2.13), is valid on $R_1$.

If $z \in R_2$, then

$$[\Delta(z)]^{\pm 1} = (\sinh a)^{\pm 2}\, O(1) = [\tfrac{1}{2} + \tfrac{1}{4}\lambda'/h^2]^{\pm 1}\, O(1), \tag{2.17}$$

and (2.14) follows readily from (1.7*b*) in its original form, subject to the condition (1.4*b*); the indefinite integral in (2.14*a*) is then chosen to vanish at $z = 0$, to provide an effective formula for estimating the variation of the e.c.f. along paths issuing from $z = 0$. This establishes part (ii) of the set of formulae; part (iii) is proved similarly.

The following lemma, valid for the parameter range under consideration, will be needed later.

Lemma 1. *If $z$, $z^* \in R_2$, and if $\xi^*$ is the value of $\xi$ at $z^*$, then, uniformly with respect to $z$, $z^*$ and the parameters,*

$$\text{(i)}\quad (\xi^* - \xi_0)^{-1} = \begin{cases} [\Delta(z)]^{-1}\, O(1) & \text{if } \Delta(0) \text{ is bounded,} \\ [\Delta(z)]^{-\frac{1}{2}}\, O(1) & \text{if } [\Delta(0)]^{-1} \text{ is bounded;} \end{cases} \tag{2.18}$$

$$\text{(ii)}\quad \Psi_2(z) = [\xi^* - \xi_0]^{-1}\, O(1). \tag{2.19}$$

*A similar result holds with $R_3$ in place of $R_2$.*

*Proof.* It follows immediately from (2.1) or (2.8), according to the value of $\lambda'/h^2$, and from the definition of $R_2$ that

$$(\xi^* - \xi_0)^{-1} = \begin{cases} [\Delta(z^*)]^{-1}\, O(1) & \text{if } \Delta(z^*) \text{ is bounded,} \\ [\Delta(z^*)]^{-\frac{1}{2}}\, O(1) & \text{if } [\Delta(z^*)]^{-1} \text{ is bounded.} \end{cases}$$

The use of (2.17) then establishes (2.18), since $\Delta(0) = -\sinh^2 a$.

Next, it is easy to see with the aid of (2.17) that

$$\Psi_2(z) = 2\mathrm{i}\, \mathrm{e}^{-\mathrm{i}z} \sin z\, [\Delta(0)]^{-\frac{3}{2}} = \begin{cases} [\Delta(0)]^{-1}\, O(1) & \text{if } \Delta(0) \text{ is bounded,} \\ [\Delta(0)]^{-\frac{1}{2}}\, O(1) & \text{if } [\Delta(0)]^{-1} \text{ is bounded,} \end{cases}$$

for on $R_2$, $\mathrm{e}^{-\mathrm{i}z}$ is bounded if $\Delta(0) = \sinh^2 a$ is bounded, and is equal to $[\Delta(0)]^{\frac{1}{2}}\, O(1)$ if $[\Delta(0)]^{-1}$ is bounded. Again by using (2.17), followed by an application of (2.1) or (2.8),

$$\Psi_2(z) = \begin{cases} [\Delta(z^*)]^{-1}\, O(1) & \text{if } \Delta(z^*) \text{ is bounded,} \\ [\Delta(z^*)]^{-\frac{1}{2}}\, O(1) & \text{if } [\Delta(z^*)]^{-1} \text{ is bounded,} \end{cases}$$
$$= [\xi^* - \xi_0]^{-1}\, O(1)$$

as required. The proof for $R_3$, with $\Psi_3$, $\Delta(\frac{1}{2}\pi)$ in place of $\Psi_2$, $\Delta(0)$, is similar.

## 3. The variation of the e.c.f. on specified paths; $\lambda' \leqslant -2h^2$

It is not essentially difficult, but is by no means trivial, to show that there exist paths of the following classes, satisfying the conditions of lemma 1 of part III, §5 and also of its corollary, the point $z^*$ in the latter being the terminal point of the path unless otherwise specified. This can be done by using paths on which either Re $\xi$ or Im $\xi$ is constant, or composed of two subarcs with this property; specimen paths of classes A and C are illustrated in figure 2. In the following description, $R_j^*$ $(j = 1, 2)$ is the union of $R_j$ and its reflexion in the imaginary axis.

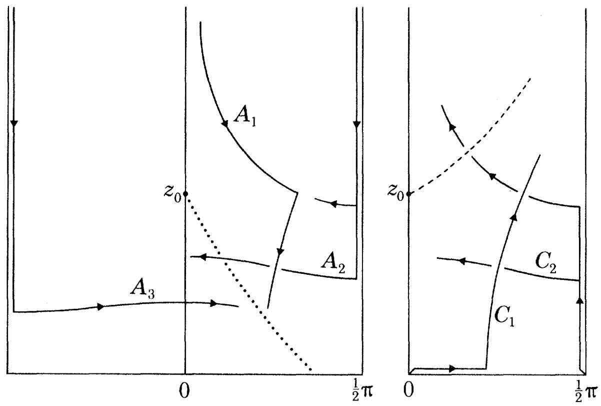


Figure 2. Progressive paths of types A and C (——►——). · · ·, arg $(\xi-\xi_0) = -\frac{1}{4}\pi$; - - -, arg $(\xi-\xi_0) = \frac{1}{2}\pi$.

*Class* A: paths originating from infinity and terminating in $\Omega$, comprising an arc in $R_1^*$, followed by an arc in $R_2^*$ if the terminal point is not in $R_1$; in the latter case, $z^*$ is the common end-point of these two arcs. This class is to comprise three subclasses:

A 1, $+\xi$-progressive paths with Im $\xi$ = const. on an initial segment, lying in and terminating at an arbitrary point of the region $\{z \in \Omega\colon \arg(\xi-\xi_0) \geqslant -\frac{1}{4}\pi\}$;

A 2, $-\xi$-progressive paths with Re $z = \frac{1}{2}\pi$ on an initial segment, lying in and terminating at an arbitrary point of $\Omega$;

A 3, $+\xi$-progressive paths with Re $z = -\frac{1}{2}\pi$ on an initial segment and terminating at an arbitrary point of $\{z \in \Omega\colon \arg(\xi-\xi_0) \leqslant -\frac{1}{4}\pi\}$; the path must enter $\Omega$ across the segment $[0, z_0)$ and $\xi$ is defined by continuation across this segment.

The two subregions of $\Omega$ given are complementary apart from having a common frontier, the second being empty if $\lambda' = -2h^2$.

On paths $\gamma$ of any of these subclasses, with terminal point $z \in \Omega$, the following holds.

Formula A

$$\operatorname*{var}_{\gamma} \int \psi(z)\,\mathrm{d}\xi = O(1) \times \begin{cases} \sinh 2a\,[\Delta(z)]^{-\frac{3}{2}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{2}\sinh^2 a, \\ [\Delta(z)]^{-1} & \text{if} \quad \frac{1}{2}\sinh^2 a \leqslant |\Delta(z)| \leqslant 1, \\ [\Delta(z)]^{-\frac{1}{2}} & \text{if} \quad |\Delta(z)| \geqslant \max\{1, \frac{1}{2}\sinh^2 a\}. \end{cases} \tag{3.1}$$

The second case only arises if $-4h^2 \leqslant \lambda' \leqslant -2h^2$. The various subdomains of $\Omega$ determined by the inequalities are illustrated in figure 1.

If $z \in R_1$, this formula follows immediately from (2.1) or (2.8), (2.13$a$) and the corollary to lemma 1 of part III. If $z \notin R_1$, then $\lambda' < -4h^2$ and by means of lemma 1 it can be shown that the estimate $[\Delta(z)^*]^{-\frac{1}{4}}\, O(1)$ applies to both subarcs and therefore to the whole, and that this estimate remains valid if $z$ is substituted for $z^*$. Some restriction on arg $(\xi-\xi_0)$ in defining domains of accessibility by paths of classes A 1 and A 3 is required in order that the condition of the corollary to lemma 1 of part III may be satisfied and hence that the variation of the e.c.f. may be expressed as a function of the terminal point; the common bound $-\frac{1}{4}\pi$ could however be replaced by $-\frac{1}{2}\pi+\delta$ and $-\delta$ $(\delta > 0)$ for classes A 1 and A 3 respectively, giving some overlap of the domains.

Next, there are two paths with terminal point at infinity:

B 1, a path of class A 2 terminating at $z = 0$, together with its reflexion in the imaginary axis, the former described in the reverse sense, so that the complete path originates from $-\frac{1}{2}\pi+\infty\,\mathrm{i}$, terminates at $\frac{1}{2}\pi+\infty\,\mathrm{i}$ and is $+\xi$-progressive;

B 2, a path of class A 2 terminating at $z = \frac{1}{2}\pi$, together with its reflexion in the real axis, the former described in the reverse sense; both Re $z$ and Re $\xi$ are constant on this path, which originates from $\frac{1}{2}\pi-\infty\,\mathrm{i}$ and terminates at $\frac{1}{2}\pi+\infty\,\mathrm{i}$.

The variation along either of these paths is twice that along the first part of class A 2. Hence by formula A the following holds.

FORMULA B

$$\text{B1: } \underset{\text{B1}}{\operatorname{var}} \int \psi(z)\, \mathrm{d}\xi = O(1) \times \begin{cases} (\sinh a)^{-2} & \text{if} \quad \sinh^2 a \leqslant \frac{1}{2} \\ (\sinh a)^{-1} & \text{if} \quad \sinh^2 a \geqslant \frac{1}{2} \end{cases} \tag{3.2$a$}$$

$$\text{B2: } \underset{\text{B2}}{\operatorname{var}} \int \psi(z)\, \mathrm{d}\xi = (\cosh a)^{-1}\, O(1). \tag{3.2$b$}$$

Lastly, there are two classes with finite initial point.

*Class* C 1: paths originating from $z = 0$, $+\xi$-progressive and terminating at an arbitrary point of $\{z \in \Omega\colon \arg(\xi-\xi_0) \leqslant \frac{1}{2}\pi-\delta\}$ $(\delta > 0)$. If the terminal point is in $R_2$, the path is to lie in $R_2$. Otherwise it is to consist of two subarcs, the first in $R_2$ and the second not meeting $R_2$ except at its initial point.

*Class* C 2: paths originating from $z = \frac{1}{2}\pi$, $-\xi$-progressive and terminating at an arbitrary point of $\{z \in \Omega\colon \arg(\xi-\xi_0) \leqslant \pi-\delta\}$ $(\delta > 0)$. The description is similar, but with $R_3$ in place of $R_2$.

By similar means to those employed for class A, the following can be established.

FORMULA C

(i) If $\gamma$ is of class C 1,

$$\underset{\gamma}{\operatorname{var}} \int \psi(z)\, \mathrm{d}\xi = O(1) \times \begin{cases} (\mathrm{e}^{-2\mathrm{i}z}-1)\, [\Delta(0)]^{-\frac{3}{4}} & \text{if} \quad z \in R_2, \\ \cosh a\, (\sinh a)^{-2} & \text{if} \quad |\Delta(z)| \geqslant \frac{1}{2}\sinh^2 a \quad \text{and} \quad z \notin R_2, \end{cases} \tag{3.3$a$}$$

and satisfies (3.1) otherwise.

(ii) If $\gamma$ is of class C 2,

$$\underset{\gamma}{\operatorname{var}} \int \psi(z)\, \mathrm{d}\xi = O(1) \times \begin{cases} (\mathrm{e}^{-2\mathrm{i}z}+1)\, [\Delta(\frac{1}{2}\pi)]^{-\frac{3}{4}} & \text{if} \quad z \in R_3, \\ (\cosh a)^{-1} & \text{if} \quad |\Delta(z)| \geqslant \frac{1}{2}\cosh^2 a \quad \text{and} \quad z \notin R_3, \end{cases} \tag{3.3$b$}$$

and satisfies (3.1) otherwise.

## 4. Asymptotic formulae for Mathieu functions; $\lambda' \leqslant -2h^2$

It is convenient to introduce, for the purpose of constructing continuations of $\xi$ into contiguous fundamental regions, the notation $\xi(z)$ for the value of $\xi$ on the branch already defined, continued into the region $\{z\colon 0 \leqslant \operatorname{Re} z \leqslant \pi, \operatorname{Im} z \geqslant 0\}$. Two real auxiliary quantities are also required, in addition to the parameters $(\mu, \Phi)$; these are

$$E = \operatorname{Im} \xi(0), \quad E_1 = -\xi_0. \tag{4.1}$$

By referring to figure 3, it is seen that

$$E = \mathrm{i}(\xi_0 - \xi(0)) < 0, \quad E_1 = \xi(\tfrac{1}{2}\pi) - \xi(0) > 0. \tag{4.1a}$$

These quantities are complete elliptic integrals of the second kind; expressions in terms of $z$ and of standard elliptic integrals, as well as approximate formulae derived from known expansions, appear in §6.1.

The remainders of the formulae of §6 below are expressed in the form III, (1.5$b$), the variation of the e.c.f. being estimated by means of the formulae of §3 above; for remarks on the validity of these remainder terms, see part I, §2($b$). For formulae in terms of $\mathrm{e}^{\pm h\xi}$, the fact that the absolute value of the exponential function is taken as majorant results in a slight simplification of form; the same is true where the basic function is a hyperbolic function of a *real* variable, but where it is a circular function of a real variable, the appropriate majorant is the unit constant function. The formulae are all valid on $\Omega$, on a specified subdomain of $\Omega$ or on a specified interval on its frontier; in some cases they are in fact valid on a more extended region, with the same or a modified remainder term.

### ($a$) *Complex basis*

The formulae (6.3.1$a$, $b$) are obtained by using paths of classes A 1, A 2 respectively, the constant factors $\mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}$ having already been determined in deriving (1.9) and (1.10). A formula for $y_1(z+\pi)$, not in a convenient form, can now be written down by substitution in (6.3.1$a$):

$$y_1(z+\pi) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z+\pi) \exp\left[h\xi(z+\pi)\right](1+\eta), \tag{4.2}$$

which is valid on the half-strip

$$\{z\colon -\tfrac{1}{2}\pi \leqslant \operatorname{Re} z \leqslant 0, \operatorname{Im} z \geqslant 0\},$$

and may be extended into a subdomain of $\Omega$ by continuation across $[0, z_0)$ with the use of paths of class A 3.

On this half-strip let $\xi$, $F$ be defined by continuation from $\Omega$ across the same interval. Now for any branch of $\xi$ on the half-strip it follows from (1.3) that

$$\xi = \pm\,\xi(z+\pi) + \text{const.},$$

and the sign and constant can be determined by comparing the values of $\xi(z)$, $\xi(z+\pi)$ when $z \in [0, z_0)$. It can be seen from the form of the map $z \to \xi(z)$ (figure 3) that in fact

$$\xi(z+\pi) = \xi + 2E_1; \tag{4.3}$$

also, from the definition of the function $F$ (1.7$a$), $F(\pi) = F(0)$, both being real and positive, whence $F(z+\pi) = F(z)$. The formula

$$y_1(z+\pi) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\,\mathrm{e}^{2hE_1} F(z)\,\mathrm{e}^{h\xi}(1+\eta), \tag{4.4}$$

valid on $\{z \in \Omega: \arg(\xi-\xi_0) \leqslant -\frac{1}{4}\pi\}$ with $\eta$ given by (6.3.2), follows from (4.2) by means of these substitutions.

(*b*) *Connection coefficients*

As $z \to \frac{1}{2}\pi + \infty\,\mathrm{i}$ with $\operatorname{Re} z = \frac{1}{2}\pi$ the following hold.

(i) It follows by the use of path B 1 that

$$y_1(z+\pi) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\,\mathrm{e}^{2hE_1}\mathrm{e}^{h\xi}(1+\eta_1),$$

where by (3.2*a*)

$$\eta_1 = h^{-1}O(1) \times \begin{cases} (\sinh a)^{-1} & \text{if} \quad \lambda' \leqslant -4h^2, \\ (\sinh a)^{-2} & \text{if} \quad -4h^2 \leqslant \lambda' \leqslant -2h^2, \end{cases}$$

the principal term being identical with that in (4.4);

(ii) By (6.3.1*a*, *b*),

$$y_1(z) \sim \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\,\mathrm{e}^{h\xi}, \tag{4.5a}$$

$$y_1(z-\pi) \sim \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}F(z)\,\mathrm{e}^{-h\xi}. \tag{4.5b}$$

By using II (4.1.2) and the method of III §4(*a*), (6.2.1) follows from (4.4), (4.5*a*, *b*).

To derive formula (6.2.2), the path B 2 is used. It is necessary to construct the continuations of $\xi$, $F$ across $[0, \frac{1}{2}\pi]$ into the half-strip $\{z: 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\pi, \operatorname{Im} z \leqslant 0\}$; the formula corresponding to (4.3) is

$$\xi(\pi - z) = -\xi + 2\mathrm{i}E,$$

and since $\arg F(\frac{1}{2}\pi) = 0$, $F(\pi - z) = F(z)$. The conclusion is that, as $z \to \frac{1}{2}\pi + \infty\,\mathrm{i}$ with $\operatorname{Re} z = \frac{1}{2}\pi$,

$$y_1(-z) = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\mathrm{e}^{-2\mathrm{i}hE}F(z)\,\mathrm{e}^{h\xi}(1+\eta),$$

where by (3.2*b*), $\eta = (h\cosh a)^{-1}\,O(1) = [\lambda']^{-\frac{1}{2}}\,O(1)$. From this formula, together with (4.5*a*, *b*), and from II, (4.1.4), it follows that

$$\bar{\beta} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi}\mathrm{e}^{-2\mathrm{i}hE}(1+\eta_2),$$

where

$$\eta_2 = (\lambda')^{-\frac{1}{2}}\,O(1).$$

Hence

$$\arg\beta = \tfrac{1}{2}\pi + 2hE + 2n\pi + (\lambda')^{-\frac{1}{2}}\,O(1) \quad (n \in \mathbf{Z}); \tag{4.6}$$

by similar methods, adapted to the different parameter range, this formula can be shown to remain valid on the extended parameter range $\lambda' < 0$ (see §5.2).

It is now necessary to determine the value of $n$. There is a constant $k > 0$ such that if $\lambda' \leqslant -k$, then $|\eta_2| < 1$; it follows that the remainder term in (4.6) does not exceed $\frac{1}{2}\pi$ in absolute value, whence by continuity, $n$ is constant. Now if $\lambda = -2h^2$, then $c_0(q) < \lambda < a_0(q)$ and $0 < \arg\beta < \pi$ (see part II, §§1(*a*, *b*), 3.3(*b*), 3.5); also by a property of $E$ given in §6.1, $hE = (\lambda' + 2h^2)\,h^{-1}O(1)$, and this expression is $h^{-1}O(1)$ since $\lambda' - \lambda$ is bounded. Hence in (4.6), $n = 0$.

At this stage it is necessary to appeal to lemma 1 from part V, §3.1(*d*). With (1.4) above this shows that there is a constant $k$ such that provided $|\lambda'|^{-\frac{1}{2}} \leqslant k$, the definitions and formulae of part II, §4.2 are applicable and

$$\mathrm{e}^{-\pi\mu} = [\lambda']^{-\frac{1}{2}}\,O(1). \tag{4.7}$$

From this and II (4.2.2) it can be shown that under the same condition

$$\arg\beta - 2\Phi = (\lambda')^{-\frac{1}{2}}\,O(1);$$

(6.2.2) follows from this and (4.6).

The formula (6.3.1 $c$), which is in effect a modification of (4.4), valid on a larger domain but with the same remainder estimate (6.3.2), can now be obtained. By II (4.1.2),

$$y_1(z+\pi) = 2\cosh(\pi\mu)\, y_1(z) - y_1(z-\pi).$$

If $|\arg(\xi-\xi_0)| \leqslant \frac{1}{4}\pi$ and if $2\cosh(\pi\mu)\, y_1(z)$ is approximated by means of (6.3.1 $a$), it can be shown that the term $y_1(z-\pi)$ can be absorbed into the remainder term, whence (6.3.1 $c$) is valid on this subdomain; likewise, if $\arg(\xi-\xi_0) \leqslant -\frac{1}{4}\pi$, the discrepancy between (4.4) and (6.3.1 $c$) can be absorbed into the remainder, so that the latter is valid with remainder estimate (6.3.2) on the region $\{z \in \Omega\colon \arg(\xi-\xi_0) \leqslant \frac{1}{4}\pi\}$. The domains of validity for (6.3.1 $a$, $c$) given in §6 are smaller than those obtained here, but are adequate since they are complementary as subsets of $\Omega$. One or other of these two, together with (6.3.1 $b$), always provides a satisfactory basis.

If $\operatorname{Re} z = \frac{1}{2}\pi$, $y_1(z)$ and $y_1(z-\pi)$ are complex conjugates; also, by (4.7), $\operatorname{sech}(\pi\mu) = (\lambda')^{-\frac{1}{4}} O(1)$, so that by using the same method, (6.3.1 $c$) is valid with the weaker estimate

$$\eta = (\lambda')^{-\frac{1}{4}}\, O(1). \tag{4.8}$$

(*c*) *Real bases*

For the modified equation II (1.3 $a$), the chosen basis comprises $y_1(\mathrm{i}x)$ and the solution $y_2(\mathrm{i}x)$ defined in II (4.2.2). If $x > a$, the formula (6.3.1 $a$) with $z = \mathrm{i}x$ gives a formula for $y_1(\mathrm{i}x)$. If $0 \leqslant x < a$, the relation

$$y_1(\mathrm{i}x) = (\cosh \pi\mu)^{-1} \operatorname{Re} y_1(\mathrm{i}x-\pi),$$

obtained from II (4.1.2) and the symmetry properties of the differential equation, with (6.3.1 $b$), gives

$$y_1(\mathrm{i}x) = (\cosh \pi\mu)^{-1} F(z) \operatorname{Re}\{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\, \mathrm{e}^{-h\xi}(1+\eta)\}, \tag{4.9a}$$

the factor $F(z)$ being real and positive. Now the remainder term in (6.2.1) does not exceed in order of magnitude the least value of the estimate (6.3.2) for $\eta$ on $[0, a)$; hence the factor $(\cosh \pi\mu)^{-1}$ may be replaced by $2\mathrm{e}^{-2hE_1}$, the effect being absorbed into the remainder term.

Similarly by II (4.2.2), for all $x \geqslant 0$,

$$\begin{aligned} y_2(\mathrm{i}x) &= -(\sinh \pi\mu)^{-1} \operatorname{Im} y_1(\mathrm{i}x-\pi) \\ &= -(\sinh \pi\mu)^{-1} \operatorname{Im}\{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi} F(z)\, \mathrm{e}^{-h\xi}(1+\eta)\}. \end{aligned} \tag{4.9b}$$

Now by (4.7), $\coth(\pi\mu) = 1 + [\lambda']^{-\frac{1}{4}} O(1)$, whence it can be seen that, if $x \in [0, a)$ but not otherwise, the factor $(\sinh \pi\mu)^{-1}$ can also be replaced by $2\mathrm{e}^{-2hE_1}$. The formulae (6.3.4) can be obtained from (4.9 $a$, $b$) thus simplified, while the second of (6.3.3) follows from (4.9 $b$) in its original form.

For the ordinary equations II, (1.2 $a$, $b$) a basis is provided by the characteristic solutions; II, (3.14) and II, (3.16) give, with $z = x$,

$$\mathrm{me}(\pm x) = \pm(\sinh 2\pi\mu)^{-1}\, \mathrm{e}^{\mp \mathrm{i}\Phi}\{\mathrm{e}^{\pm\pi\mu} y_1(z+\pi) - \mathrm{e}^{\mp\pi\mu} y_1(z-\pi)\}.$$

By (6.3.1 $b$, $c$),

$$\mathrm{e}^{\pm\pi\mu} y_1(z+\pi) - \mathrm{e}^{\mp\pi\mu} y_1(z-\pi) = F(z)\{2\cosh(\pi\mu)\, \mathrm{e}^{\pm\pi\mu}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\mathrm{e}^{h\xi}(1+\eta_1) - \mathrm{e}^{\mp\pi\mu}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\mathrm{e}^{-h\xi}(1+\eta_2)\},$$

where $\eta_1$, $\eta_2$ both satisfy estimate (6.3.2).

By means of (4.7) it can be shown that on $[0, \frac{1}{2}\pi]$ the term with exponent $+\pi\mu$ dominates by a factor of the order of $e^{\pi\mu}$ at least, and that the smaller term can be absorbed into the remainder of the larger term. The result is:

$$\mathrm{me}\,(x) = (\sinh 2\pi\mu)^{-1}\, e^{i\Phi} e^{\pi\mu} \cosh(\pi\mu)\, e^{\frac{1}{4}i\pi} F(z)\, e^{h\xi}(1+\eta),$$

$$\mathrm{me}\,(-x) = (\sinh 2\pi\mu)^{-1}\, e^{-i\Phi} e^{\pi\mu} e^{-\frac{1}{4}i\pi} F(z)\, e^{-h\xi}(1+\eta).$$

The formulae (6.3.5) and (6.3.7) are now derived from these by means of (4.7) and (6.2.2), the discrepancies being again absorbed into the remainder, which has estimate (6.3.6) obtained from (6.3.2) by restriction to $z \in [0, \frac{1}{2}\pi]$.

The formula $$\mathrm{Me}^*\,(\pm x) = 2F(z)\, e^{\pm h(\xi - iE)}(1+\eta),$$

where $z = \frac{1}{2}\pi + ix$ and $\eta = (h\cosh a)^{-1}\, O(1) = [\lambda']^{-\frac{1}{4}}\, O(1)$, can be obtained similarly; here (6.3.1$c$) has remainder estimate (4.8) and this is the dominant contribution. From this formula and II, (4.2.11), there follows (6.3.8).

Except for (6.3.8), formulae for even and odd functions are not included in §6; they may be obtained by means of formulae given in II, §4 expressing them in terms of the appropriate basis. Alternative formulae, which have the same principal term apart from a factor independent of $x$ or $z$ but have a sharper remainder estimate under some conditions, can be derived by using paths of class C 1 or C 2 as appropriate; they do not depend on any specific normalization. A typical pair of formulae is

$$\mathrm{ce}\,(z) = \mathrm{ce}\,(0)\,[F(0)]^{-1} F(z) \cosh[h(\xi - \xi(0))]\,(1+\eta)$$

$$\mathrm{se}\,(z) = \mathrm{se}'\,(0)\, F(0)\, h^{-1} F(z) \sinh[h(\xi - \xi(0))]\,(1+\eta),$$

where $\eta = h^{-1}\,\mathrm{var} \int \psi(z)\, d\xi\, O(1)$, the variation satisfying (3.3$a$) on the region accessible by paths of class C 1.

## 5. Other parameter ranges

### 5.1. *The parameter range* $\lambda' \geqslant 2h^2$

For this range, the differential equation has just one transition point in $\Omega$, on its frontier at $z_0 = \frac{1}{2}\pi + ia$, where $a > 0$ satisfies $\lambda' - 2h^2\cosh 2a = 0$; let $\xi_0$ again be the value of $\xi(z_0)$. The calculations leading to the formulae of §2($c$) above require only minor modification, principally the substitution of $\cos z$ for $\sin z$; the formulae themselves, as well as the formulae (2.1) and (2.8), do not require modification. The construction of progressive paths is based on similar principles, though there are differences in detail, and the final formulae for the variation of the e.c.f., as given in §3, remain unchanged.

The types of path to which the various formulae are applicable are as follows:

*Formula* A. Paths which terminate at a point of $\Omega$, and which:

($a$) originate from $+\infty i$, and are $+\xi$-progressive;

($b$) originate from $\frac{1}{2}\pi + \infty i$, and are $-\xi$-progressive, validity being restricted to $\{z \in \Omega: \arg(\xi - \xi_0) \leqslant \frac{5}{4}\pi\}$;

($c$) originate from $-\infty i$, enter $\Omega$ across $[0, \frac{1}{2}\pi]$, and are $-\xi$-progressive, validity being restricted to $\{z \in \Omega: \arg(\xi - \xi_0) \geqslant \frac{5}{4}\pi\}$.

*Formula* B 1. A path which originates from $\pi - \infty i$, is $-\xi$-progressive and terminates at $+\infty i$, with centre point $z = \frac{1}{2}\pi$.

*Formula* B 2. A path which originates from $-\infty\,\mathrm{i}$, is $-\xi$-progressive and terminates at $\infty\,\mathrm{i}$, with centre point $z = 0$.

*Formulae* C. Paths which originate from $z = 0$ or $z = \frac{1}{2}\pi$, and terminate at a point of the sub-domain $\{z \in \Omega\colon \arg(\xi-\xi_0) \geqslant \frac{3}{4}\pi$ or $\frac{5}{4}\pi$ respectively$\}$.

The methods used to derive the various asymptotic formulae given in §6, both for solutions and for connection formulae, are similar to the methods of §4. The two real auxiliary quantities are now

$$E = \operatorname{Im}\xi_0 > 0, \quad E_1 = -\xi(0) = \xi_0 - \xi(\tfrac{1}{2}\pi) \leqslant 0.$$

## 5.2. *The parameter range* $-2h^2 \leqslant \lambda' \leqslant 2h^2$

The differential equation now has one transition point in $\Omega$, on its frontier at $z_0 = a$ where $a \in [0, \frac{1}{2}\pi]$ and $\lambda' + 2h^2 \cos 2a = 0$; let $\xi_0 = \xi(z_0)$ and let

$$E = \operatorname{Im}\xi_0 \geqslant 0, \quad E_1 = -\operatorname{Re}\xi_0 \geqslant 0.$$

The estimates of §2(*c*) are again valid, but $R_1 = \Omega$ over the whole range. The derivation when $-2h^2 \leqslant \lambda' \leqslant 0$ is a modification of that used for the case $-4h^2 \leqslant \lambda' \leqslant -2h^2$, the main difference being the substitution of circular for hyperbolic functions; when $0 \leqslant \lambda' \leqslant 2h^2$, the cosine function must be substituted for the sine function throughout. A formula corresponding to (2.8) is

$$(\xi-\xi_0)^{\pm 1} = O(1) \times \begin{cases} (\sin 2a)^{\mp 1}\,[\Delta(z)]^{\pm\frac{3}{4}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a, \\ [\Delta(z)]^{\pm 1} & \text{if} \quad \frac{1}{8}\sin^2 2a \leqslant |\Delta(z)| \leqslant 1, \\ [\Delta(z)]^{\pm\frac{1}{2}} & \text{if} \quad |\Delta(z)| \geqslant 1; \end{cases}$$

the essential difference from (2.8) lies in the expression $\frac{1}{8}\sin^2 2a$, whose significance is that it is of the order of $\frac{1}{2}\sin^2 a$ if $\sin^2 a \leqslant \frac{1}{2}(\lambda' \leqslant 0)$, but is of the order of $\frac{1}{2}\cos^2 a$ if $\sin^2 a \geqslant \frac{1}{2}$.

The description of the classes of path is the same as in §5.1 above, except that the frontiers of the subdomains are different; for each subdomain the frontier is now defined by $\arg(\xi-\xi_0) = \frac{3}{4}\pi$. The formulae corresponding to those of §3 are:

Formula A*

$$\operatorname*{var}_{\gamma}\int \psi(z)\,\mathrm{d}\xi = O(1) \times \begin{cases} \sin 2a[\Delta(z)]^{-\frac{3}{4}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a, \\ [\Delta(z)]^{-1} & \text{if} \quad \frac{1}{8}\sin^2 2a \leqslant |\Delta(z)| \leqslant 1, \\ [\Delta(z)]^{-\frac{1}{2}} & \text{if} \quad |\Delta(z)| \geqslant 1. \end{cases} \tag{5.1}$$

Formula B*

$$\text{B 1:} \operatorname*{var}_{\text{B 1}}\int \psi(z)\,\mathrm{d}\xi = (\cos a)^{-2}\,O(1)$$

$$\text{B 2:} \operatorname*{var}_{\text{B 2}}\int \psi(z)\,\mathrm{d}\xi = (\sin a)^{-2}\,O(1).$$

Formula C*

(i) For a path of class C 1,

$$\operatorname*{var}_{\gamma}\int \psi(z)\,\mathrm{d}\xi = O(1) \times \begin{cases} \sin z\,(\sin a)^{-3} & \text{if} \quad z \in R_2, \\ (\sin a)^{-2} & \text{if} \quad |\Delta(z)| \geqslant \frac{1}{8}\sin^2 2a \quad \text{and} \quad z \notin R_2; \end{cases}$$

(ii) For a path of class C 2, substitute cosine for sine and $R_3$ for $R_2$. In each case the variation satisfies the first of (5.1) if $|\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a$.

The asymptotic formulae for solutions and connection formulae are again derived by similar methods.

## 6. Tables of asymptotic formulae

### 6.1. *Definitions*

The quantity $\lambda'$ is a redundant parameter which is required to satisfy

$$|\lambda'-\lambda| \leqslant k,$$

$$|\lambda'-\lambda| \leqslant k|h^2/\lambda'|,$$

where $k$ is arbitrary but fixed. In this section it is natural to take $\lambda' = \lambda$, but in part V, §3.4, relating to parabolic cylinder functions, $\lambda'-\lambda = \frac{1}{8}$.

Table 1. The elliptic integrals $E$, $E_1$

| | $hE$ | $hE_1$ |
|---|---|---|
| $\lambda' \leqslant -2h^2$ | $-\int_0^a \|\lambda'+2h^2\cosh 2x\|^{\frac{1}{2}}\mathrm{d}x$ $= -2h\sinh a\tanh a\,\boldsymbol{D}(\tanh a)$ $= -2h\cosh a\,\{\boldsymbol{K}(\tanh a)-\boldsymbol{E}(\tanh a)\}$ | $\int_0^{\frac{1}{2}\pi} \|\lambda'+2h^2\cos 2x\|^{\frac{1}{2}}\mathrm{d}x$ $= 2h\cosh a\,\boldsymbol{E}(\operatorname{sech} a)$ |
| | ($a > 0$ such that $\lambda' = -2h^2\cosh 2a$) | |
| $-2h^2 < \lambda' < 2h^2$ | $\int_0^a (\lambda'+2h^2\cos 2x)^{\frac{1}{2}}\mathrm{d}x$ $= 2h\sin^2 a\,\boldsymbol{B}(\sin a)$ $= 2h\{\boldsymbol{E}(\sin a)-\cos^2 a\,\boldsymbol{K}(\sin a)\}$ | $\int_0^{\frac{1}{2}\pi} \|\lambda'+2h^2\cos 2x\|^{\frac{1}{2}}\mathrm{d}x$ $= 2h\cos^2 a\,\boldsymbol{B}(\cos a)$ $= 2h\{\boldsymbol{E}(\cos a)-\sin^2 a\,\boldsymbol{K}(\cos a)\}$ |
| | ($a \in [0, \frac{1}{2}\pi]$ such that $\lambda' = -2h^2\cos 2a$) | |
| $\lambda' \geqslant 2h^2$ | $\int_0^{\frac{1}{2}\pi} (\lambda'+2h^2\cos 2x)^{\frac{1}{2}}\mathrm{d}x$ $= 2h\cosh a\,\boldsymbol{E}(\operatorname{sech} a)$ | $-\int_0^a (\lambda'-2h^2\cosh 2x)^{\frac{1}{2}}\mathrm{d}x$ $= -2h\sinh a\tanh a\,\boldsymbol{D}(\tanh a)$ $= -2h\cosh a\,\{\boldsymbol{K}(\tanh a)-\boldsymbol{E}(\tanh a)\}$ |
| | ($a > 0$ such that $\lambda' = 2h^2\cosh 2a$) | |

With this preliminary, the following are defined:

(i) the fundamental region $\Omega = \{z\colon 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\pi, \operatorname{Im} z \geqslant 0\}$,

(ii) $$\Delta(z) = -\tfrac{1}{4}h^{-2}(\lambda'+2h^2\cos 2z) = \sin^2 z-\sin^2 z_0,$$

where $z_0$ is the transition point (zero of $\Delta(z)$) which lies on the frontier of $\Omega$,

(iii) $$F(z) = [4\Delta(z)]^{-\frac{1}{4}},$$

regular on $\Omega$ except at $z_0$, with $\arg F(z) = 0$ when $\Delta(z)$ is real and positive; finally,
(iv) $\xi$ is defined on $\Omega$ by

$$\mathrm{d}\xi/\mathrm{d}z = 2[\Delta(z)]^{\frac{1}{2}},$$

the branch and constant of integration being determined as follows. If $\operatorname{Re} z = \frac{1}{2}\pi$, then $\arg\xi = \frac{1}{2}\pi$ and if $\operatorname{Re} z = 0$, then $\arg\xi = \pi$, both for all sufficiently large values of $\operatorname{Im} z$; $\xi_0$ denotes the value of $\xi$ where $\lambda'+2h^2\cos 2z = 0$. The map $z \to \xi$ is one-to-one on $\Omega$; it is illustrated in different cases in figures 3–5 (see the following subsections).

The expression $O(1)$ represents throughout a function of the relevant quantities which is *bounded* under the conditions specified in each case; fuller remarks on the interpretation of remainder terms appear in part I, §2(*b*). Definitions of the elliptic integrals $E$, $E_1$ and the associated quantity $a$, with formulae in terms of the standard integrals $\boldsymbol{K}$, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ (Erdélyi *et al.* 1953), are given in table 1; they are monotone functions of $\frac{1}{2}\lambda'/h^2$ and in critical ranges of this variable, $E$, $E_1$ have the following approximate representations.

(i) $\frac{1}{2}\lambda'/h^2 \approx 1$:

$$E_1(\tfrac{1}{2}\lambda'/h^2) = E(-\tfrac{1}{2}\lambda'/h^2) = \tfrac{1}{2}\pi(k^2+\tfrac{1}{8}k^4+\ldots),$$

$$E(\tfrac{1}{2}\lambda'/h^2) = E_1(-\tfrac{1}{2}\lambda'/h^2) = 2\{1-k^2(\tfrac{1}{2}\ln[4/|k|]+\tfrac{1}{4})+o(k^2)\},$$

where $k^2 = \frac{1}{2}-\lambda'/h^2$.

(ii) $\frac{1}{2}\lambda'/h^2 \to +\infty$:

$$E_1(\tfrac{1}{2}\lambda'/h^2) = E(-\tfrac{1}{2}\lambda'/h^2) = -2\sinh a\tanh a\{\ln(4\cosh a)-\tanh^2 a+o(1)\},$$

$$E(\tfrac{1}{2}\lambda'/h^2) = E_1(-\tfrac{1}{2}\lambda'/h^2) = \pi\cosh a\{1-\tfrac{1}{4}\operatorname{sech}^2 a+O(\operatorname{sech}^4 a)\}.$$

## 6.2. *Auxiliary parameters*

(*a*) *The range* $\lambda' \leqslant -2h^2$

The following formulae and those of §6.3 below are based on the use of the definitions of part II, §4.2; there is a constant $k$ such that this is justified if $|\lambda'| > k$:

$$2\cosh(\pi\mu) = e^{2hE_1}(1+\eta_1), \tag{6.2.1}$$

where
$$\eta_1 = \begin{cases} h^{-1}(\sinh a)^{-2}\,O(1) & \text{if} \quad -4h^2 \leqslant \lambda' \leqslant -2h^2, \\ h^{-1}(\sinh a)^{-1}\,O(1) & \text{if} \quad \lambda' \leqslant -4h^2; \end{cases}$$

$$\Phi = \tfrac{1}{4}\pi+hE+\eta_2, \tag{6.2.2}$$

where
$$\eta_2 = (h\cosh a)^{-1}\,O(1) = [\lambda']^{-\frac{1}{2}}\,O(1).$$

(*b*) *The range* $\lambda' \geqslant -2h^2$

The following formulae and those of §§6.4, 6.5 below are based on the use of the definitions of part II, §4.3; there is a constant $k$ such that this is justified if *either* $h > k$ *or* $|\lambda'| > k$:

$$\hat{\beta} = e^{-2hE_1}(1+\eta_1), \tag{6.2.3}$$

where
$$\eta_1 = \begin{cases} [h\Delta(0)]^{-1}\,O(1) & \text{if} \quad \lambda' \leqslant 4h^2, \\ (h\cosh a)^{-1}\,O(1) & \text{if} \quad \lambda' \geqslant 4h^2; \end{cases}$$

$$\Phi = \tfrac{1}{4}\pi+hE+\eta_2, \tag{6.2.4}$$

where
$$\eta_2 = \begin{cases} [h\Delta(\tfrac{1}{2}\pi)]^{-1}\,O(1) & \text{if} \quad \lambda' \leqslant 4h^2, \\ (h\sinh a)^{-1}\,O(1) & \text{if} \quad \lambda' \geqslant 4h^2. \end{cases}$$

In certain cases, more refined formulae are given in part V, but with constraints on the difference $\lambda'-\lambda$.

### 6.3. *Solutions of the Mathieu equation, $\lambda' \leqslant -2h^2$*

(*a*) *Complex variable*: $y'' + (\lambda + 2h^2 \cos 2z)\, y = 0$

The following are valid on $\Omega = \{z\colon 0 \leqslant \mathrm{Re}\, z \leqslant \frac{1}{2}\pi,\ \mathrm{Im}\, z \geqslant 0\}$ or on a specified subregion:

$$y_1(z) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} F(z)\, \mathrm{e}^{h\xi}(1+\eta) \quad \text{on} \quad \{z \in \Omega\colon \mathrm{Im}\,(\xi - \xi_0) \geqslant 0\}, \tag{6.3.1a}$$

$$y_1(z-\pi) = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi} F(z)\, \mathrm{e}^{-h\xi}(1+\eta) \quad \text{on} \quad \Omega, \tag{6.3.1b}$$

$$y_1(z+\pi) = 2\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} \cosh\,(\pi\mu)\, F(z)\, \mathrm{e}^{h\xi}(1+\eta) \quad \text{on} \quad \{z \in \Omega\colon \mathrm{Im}\,(\xi - \xi_0) \leqslant 0\}, \tag{6.3.1c}$$

where in each case, uniformly,

$$\eta = h^{-1}O(1) \times \begin{cases} \sinh 2a\, [\varDelta(z)]^{-\frac{3}{2}} & \text{if} \quad |\varDelta(z)| \leqslant \frac{1}{2}\sinh^2 a \\ [\varDelta(z)]^{-1} & \text{if} \quad \frac{1}{2}\sinh^2 a \leqslant |\varDelta(z)| \leqslant 1 \\ [\varDelta(z)]^{-\frac{1}{2}} & \text{if} \quad |\varDelta(z)| \geqslant \max\,\{1, \frac{1}{2}\sinh^2 a\}; \end{cases} \tag{6.3.2}$$

the second form only arises if $\lambda' > -4h^2$. Figure 3 shows the subdomains in (6.3.1*a*, *c*).

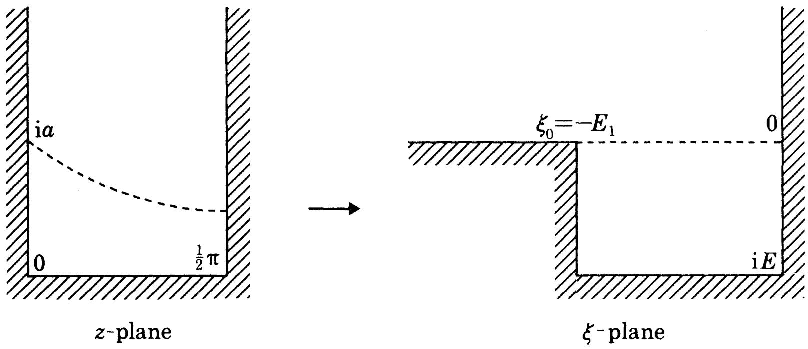


FIGURE 3. The map $z \to \xi$, $\lambda' < -2h^2$.

(*b*) *Modified equation*, $q < 0$: $y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0$

With $z = \mathrm{i}x$ the following hold.

If $x > a$, then $\xi$ is real, $\xi < -E_1$ and

$$\left.\begin{aligned} y_1(\mathrm{i}x) &= |F(z)|\, \mathrm{e}^{h\xi}(1+\eta), \\ y_2(\mathrm{i}x) &= (\sinh \pi\mu)^{-1}\, |F(z)|\, \mathrm{e}^{-h\xi}(1+\eta). \end{aligned}\right\} \tag{6.3.3}$$

If $0 \leqslant x < a$, then $F(z)$ and $\mathrm{i}(\xi + E_1)$ are both real and positive, and

$$\left.\begin{aligned} y_1(\mathrm{i}x) &= \mathrm{e}^{-hE_1} F(z)\, \{\cos\,[h|\xi + E_1| - \tfrac{1}{4}\pi] + \eta\}, \\ y_2(\mathrm{i}x) &= \mathrm{e}^{-hE_1} F(z)\, \{\cos\,[h|\xi + E_1| + \tfrac{1}{4}\pi] + \eta\}. \end{aligned}\right\} \tag{6.3.4}$$

In each case, $\eta$ satisfies (6.3.2).

(*c*) *Ordinary functions*, $q < 0$: $y'' + (\lambda + 2h^2 \cos 2x)\, y = 0$

With $z = x$ the following hold on $[0, \frac{1}{2}\pi]$:

$$\left.\begin{aligned} \mathrm{me}\,(x) &= 2F(z)\, \mathrm{e}^{h(\xi - \mathrm{i}E)}(1+\eta) \\ \mathrm{me}\,(\pi - x) &= \mathrm{e}^{\pi\mu}\, \mathrm{me}\,(-x) = 2F(z)\, \mathrm{e}^{-h(\xi - \mathrm{i}E)}(1+\eta), \end{aligned}\right\} \tag{6.3.5}$$

where, uniformly,
$$\eta = h^{-1}O(1) \times \begin{cases} [\varDelta(z)]^{-\frac{1}{2}} & \text{if} \quad |\varDelta(z)| \geqslant 1, \\ [\varDelta(z)]^{-1} & \text{otherwise}; \end{cases} \tag{6.3.6}$$

$F(z)$ is real and positive while $\xi - \mathrm{i}E$ is real and less than or equal to zero.


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(*d*) *Ordinary functions*, $q > 0: y'' + (\lambda - 2h^2 \cos 2x)\, y = 0$

With $z = x + \frac{1}{2}\pi$, the following hold if $|x| \leqslant \frac{1}{2}\pi$:

$$\mathrm{me}^*\,(\pm x) = 2F(z)\, \mathrm{e}^{\pm h(\xi - \mathrm{i}E)}(1+\eta), \tag{6.3.7}$$

where $\eta$ satisfies (6.3.6).

(*e*) *Modified functions*, $q > 0: y'' - (\lambda - 2h^2 \cosh 2x)\, y = 0$

With $z = \frac{1}{2}\pi + \mathrm{i}x$, the following hold if $x \in [0, \frac{1}{2}\pi]$:

$$\left.\begin{aligned} \mathrm{Ce}^*(x) &= 2F(z)\,\{\cos\,[h|\xi - \mathrm{i}E|] + \eta\}, \\ \mathrm{Se}^*(x) &= 2F(z)\,\{\sin\,[h|\xi - \mathrm{i}E|] + \eta\}, \end{aligned}\right\} \tag{6.3.8}$$

where $\eta = (h \cosh a)^{-1}\, O(1)$, uniformly; $F(z)$ is real and positive, while $\mathrm{i}\xi + E$ is real and less than or equal to zero.

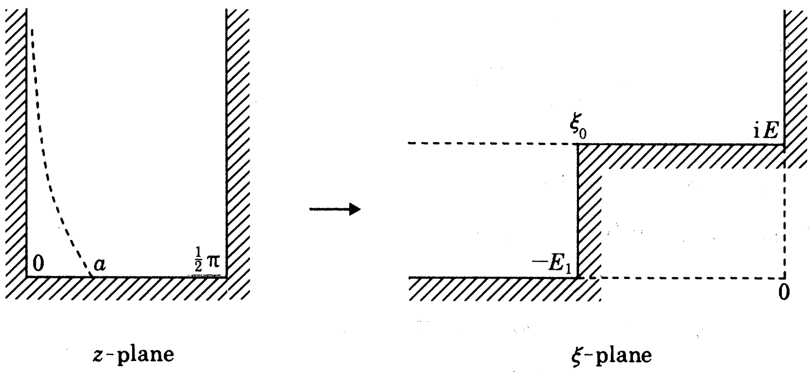


FIGURE 4. The map $z \to \xi$, $-2h^2 < \lambda' < 2h^2$.

### 6.4. *Solutions of the Mathieu equation*, $-2h^2 \leqslant \lambda' \leqslant 2h^2$

(*a*) *Complex variable*: $y'' + (\lambda + 2h^2 \cos 2z)\, y = 0$

The following are valid on $\Omega = \{z: 0 \leqslant \mathrm{Re}\, z \leqslant \frac{1}{2}\pi, \mathrm{Im}\, z \geqslant 0\}$ or on a specified subregion:

$$y_1(z) = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\, \mathrm{e}^{h\xi}(1+\eta), \quad \text{valid on } \Omega, \tag{6.4.1a}$$

$$y_1(z - \pi) = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}F(z)\, \mathrm{e}^{-h\xi}(1+\eta) \quad \text{on} \quad \{z \in \Omega: \mathrm{Im}\, \xi \geqslant E\}, \tag{6.4.1b}$$

$$y_1(-z) = \hat{\beta}\, \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}F(z)\, \mathrm{e}^{-h\xi}(1+\eta) \quad \text{on} \quad \{z \in \Omega: \mathrm{Im}\, \xi \leqslant E\}, \tag{6.4.1c}$$

where in each case, uniformly,

$$\eta = h^{-1}O(1) \times \begin{cases} \sin 2a\,[\Delta(z)]^{-\frac{3}{2}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a, \\ [\Delta(z)]^{-1} & \text{if} \quad \frac{1}{8}\sin^2 2a \leqslant |\Delta(z)| \leqslant 1 \\ [\Delta(z)]^{-\frac{1}{2}} & \text{if} \quad |\Delta(z)| \geqslant 1. \end{cases} \tag{6.4.2}$$

Figure 4 shows the subdomains in (6.4.1 *b*, *c*).

(*b*) *Modified equation*, $q < 0: y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0$

With $z = \mathrm{i}x$ the following holds.
If $x \geqslant 0$, then $\xi$ is real, $\xi < -E_1$ and

$$\left.\begin{aligned} y_1(\mathrm{i}x) &= |F(z)|\, \mathrm{e}^{h\xi}(1+\eta), \\ y_1(-\mathrm{i}x) &= \hat{\beta}|F(z)|\, \mathrm{e}^{-h\xi}(1+\eta), \end{aligned}\right\} \tag{6.4.3}$$

where, uniformly,

$$\eta = h^{-1}O(1) \times \begin{Bmatrix} [\Delta(z)]^{-\frac{1}{2}} & \text{if} \quad |\Delta(z)| \geqslant 1, \\ [\Delta(z)]^{-1} & \text{otherwise.} \end{Bmatrix} \tag{6.4.4}$$

(*c*) *Ordinary equation*, $q < 0: y'' + (\lambda + 2h^2 \cos 2x)\, y = 0$

With $z = x$ the following hold.

If $0 \leqslant x < a$, then $-\mathrm{i}(\xi + E_1)$ is real and positive and

$$\left. \begin{aligned} y_3(x) &= \mathrm{e}^{-hE_1}|F(z)|\,\{\cos\,[h|\xi-\xi_0|+\phi+\tfrac{1}{4}\pi]+\eta\}, \\ y_3(\pi - x) &= \beta^*\,\mathrm{e}^{-hE_1}|F(z)|\,\{\cos\,[h|\xi-\xi_0|+\phi-\tfrac{1}{4}\pi]+\eta\}, \end{aligned} \right\} \tag{6.4.5}$$

where $\phi = \Phi - \frac{1}{4}\pi - hE$ (see (6.2.4) and V, §§3.4, 4.4 for $\Phi$).

If $a < x \leqslant \frac{1}{2}\pi$, then $\xi - \xi_0$ is real and positive and

$$\left. \begin{aligned} y_3(x) &= \mathrm{e}^{-hE_1}F(z)\,\mathrm{e}^{h(\xi-\xi_0)}(1+\eta), \\ y_3(\pi - x) &= \mathrm{e}^{hE_1}F(z)\,\mathrm{e}^{-h(\xi-\xi_0)}(1+\eta). \end{aligned} \right\} \tag{6.4.6}$$

In each case, uniformly,

$$\eta = h^{-1}O(1) \times \begin{cases} \sin 2a\,[\Delta(z)]^{-\frac{3}{2}} & \text{if} \quad |\Delta(z)| \leqslant \frac{1}{8}\sin^2 2a, \\ [\Delta(z)]^{-1} & \text{otherwise.} \end{cases} \tag{6.4.7}$$

(*d*) *Ordinary equation*, $q > 0: y'' + (\lambda - 2h^2 \cos 2x)\, y = 0$

Formulae for $y_3(\frac{1}{2}\pi \pm x)$ valid on $[0, \frac{1}{2}\pi]$ are obtained by substituting $\frac{1}{2}\pi - x$ for $x$ in (6.4.5), (6.4.6).

(*e*) *Modified equation*, $q > 0: y'' - (\lambda - 2h^2 \cosh 2x)\, y = 0$

Formulae (6.5.1) hold on $x \geqslant 0$, with $\eta$ given by (6.4.4) and $\xi_0$ replaced by $\mathrm{i}E$.

### 6.5. *Solutions of the Mathieu equation*, $\lambda' \geqslant 2h^2$

(*a*) *Complex variable*: $y'' + (\lambda + 2h^2 \cos 2z)\, y = 0$

Formulae (6.4.1 *a*, *b*, *c*) are valid on $\Omega$ or the specified subdomain, with $\eta$ given by (6.3.2); figure 5 shows these subdomains.

(*b*) *Modified equation*, $q < 0: y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0$

Formulae (6.4.3) hold.

(*c*) *Ordinary equations*, $q < 0: y'' + (\lambda + 2h^2 \cos 2x)\, y = 0$
$q > 0: y'' + (\lambda - 2h^2 \cos 2x)\, y = 0$

Formulae (6.4.5) hold on $[0, \frac{1}{2}\pi]$ with $\eta$ given by (6.3.6) and $\xi_0$ replaced by $\mathrm{i}E - E_1$.

(*d*) *Modified equation*, $q > 0: y'' - (\lambda' - 2h^2 \cosh 2x)\, y = 0$

With $z = \frac{1}{2}\pi + \mathrm{i}x$ in the following hold.

If $x > a$, then $F(z)$ and $-\mathrm{i}(\xi - \xi_0)$ are real and positive and

$$\left. \begin{aligned} y_4(x) &= F(z)\,\{\cos\,[h|\xi-\xi_0|+\tfrac{1}{4}\pi-\phi]+\eta\}, \\ y_4(-x) &= (\hat{\beta}+|\beta|)\,F(z)\,\{\cos\,[h|\xi-\xi_0|-\tfrac{1}{4}\pi-\phi]+\eta\}, \end{aligned} \right\} \tag{6.5.1}$$

where $\phi$ is defined under (6.4.5).

If $0 \leqslant x \leqslant a$, $\xi-\xi_0$ is real and positive and

$$\left.\begin{aligned} y_4(x) &= |F(z)|\, \mathrm{e}^{h(\xi-\xi_0)}(1+\eta), \\ y_4(-x) &= \mathrm{e}^{-2hE_1}|F(z)|\, \mathrm{e}^{-h(\xi-\xi_0)}(1+\eta). \end{aligned}\right\} \tag{6.5.2}$$

In each case, $\eta$ is given by (6.3.2).

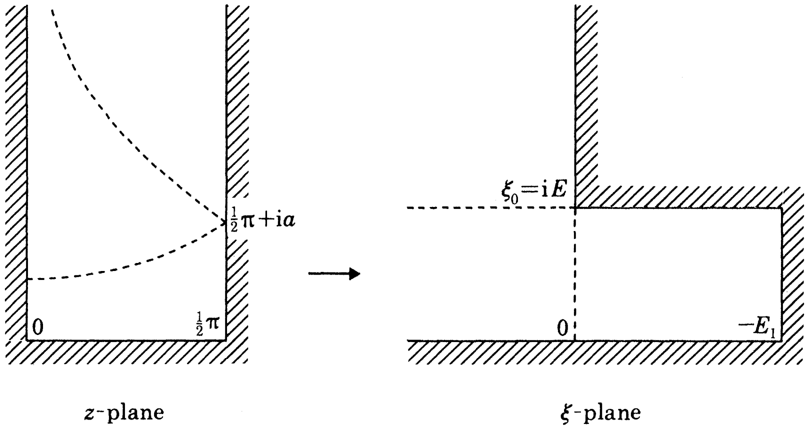


FIGURE 5. The map $z \rightarrow \xi$, $\lambda' > 2h^2$.

## REFERENCE (Part IV)

Erdélyi, A. *et al.* 1953 *Higher transcendental functions*, vol. II. New York: McGraw-Hill.



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